Topology of 3-manifolds and a class of groups II[*]

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Abstract

This is a continuation of an earlier preprint ([17]) under the same title. These papers grew out of an attempt to find a suitable finite sheeted covering of an aspherical 3-manifold so that the cover either has infinite or trivial first homology group. With this motivation we defined a new class of groups. These groups are in some sense eventually perfect. Here we prove results giving several classes of examples of groups which do (not) belong to this class. Also we prove some basic results on these groups and state two conjectures. A direct application of one of the conjectures to the virtual Betti number conjecture is mentioned. For completeness, here we reproduce parts of [17].

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0   Introduction

The main motivation to this paper and [17] came from 3-manifold topology while trying to find a suitable finite sheeted covering of an aspherical 3-manifold so that the cover has either infinite or trivial first integral homology group. In [15] it was proved that $M^3 \times \mathbb{D}^n$ is topologically rigid for $n > 1$ whenever $H_1(M^3, \mathbb{Z})$ is infinite. Also the same result is true when $H_1(M^3, \mathbb{Z})$ is 0. The remaining case is when $H_1(M^3, \mathbb{Z})$ is nontrivial finite. There are induction techniques in surgery theory which can be used to prove topological rigidity of a manifold if certain finite sheeted coverings of the manifold are also topologically rigid. In the case of manifolds with nontrivial finite first integral homology groups there is a natural finite sheeted cover, namely, the one which corresponds to the commutator subgroup of the fundamental group.

So we start with a closed aspherical 3-manifold M with nontrivial finite first integral homology group and consider the finite sheeted covering M_1 of M corresponding to the commutator subgroup. If $H^1(M_1, \mathbb{Z}) \neq 0$ or $H_1(M_1, \mathbb{Z}) = 0$ then we are done otherwise we again take the finite sheeted cover of M_1 corresponding to the commutator subgroup and continue. The group theoretic conjecture (Conjecture 0.2) in this article implies that this process stops in the sense that for some i either $H^1(M_i, \mathbb{Z}) \neq 0$ or $H_1(M_i, \mathbb{Z}) = 0$.

Motivated by the above situation we define the following class of groups.

Definition (0.0). An abstract group G is called *adorable* if $G^i/G^{i+1} = 1$ for some i, where $G^i = [G^{i-1}, G^{i-1}]$, the commutator subgroup of G^{i-1}, and $G^0 = G$. The smallest i for which the above property is satisfied is called the *degree of adorability* of G. We denote it by $doa(G)$.

Obvious examples of adorable groups are finite groups, perfect groups, simple groups and solvable groups. The second and third class of groups are adorable groups of degree 0. The free products of perfect groups are adorable (in fact perfect). The nontrivial abelian groups and symmetric groups on $n \geq 5$ letters are adorable of degree 1. Another class of adorable groups are $GL(R) = Lim_{n \to \infty} GL_n(R)$. Here R is any ring with unity and $GL_n(R)$ is the multiplicative group of $n \times n$ invertible matrices. These are adorable groups of degree 1. This follows from the Whitehead lemma which says that the commutator subgroup of $GL_n(R)$ is generated by the elementary matrices and the group generated by the elementary matrices is a perfect group. Also $SL_n(\mathbb{C})$, the multiplicative group of $n \times n$ matrices with complex entries

is a perfect group. In fact we will prove that any connected Lie group is adorable as an abstract group. The full braid groups on more than 4 strings are adorable of degree 1.

We observe the following two elementary facts in the next section.

Theorem (1.8). *A group G is adorable if and only if there is a filtration $G_n < G_{n-1} < \cdots < G_1 < G_0 = G$ of G so that G_i is normal in G_{i-1}, G_{i-1}/G_i is abelian for each i and G_n is a perfect group.*

Theorem (1.10). *Let H be a normal subgroup of an adorable group G. Then H is adorable if one of the following conditions is satisfied.*

- *G/H is solvable.*

- *for some i, G^i/H^i is abelian.*

- *for some i, G^i is simple.*

- *for some i, G^i is perfect and the group G^i/H^{i+1} does not have any proper abelian normal subgroup.*

Also the braid group on more than 4 strings are the examples to show that an arbitrary finite index normal subgroup of an adorable group need not be adorable.

In Section 4 the following result is proved about Lie groups.

Theorem (4.9). *Every connected real or complex Lie group is adorable as an abstract group.*

Below we give some examples of nonadorable groups. Proofs of nonadorability of some of these examples are easy. Proofs for the other examples are given in the next sections.

Some examples of groups which are not adorable are nonabelian free groups and fundamental groups of surfaces of genus greater than 1; for the intersection of a monotonically decreasing sequence of characteristic subgroups of a nonabelian free group consists of the trivial element only. The commutator subgroup of $SL_2(\mathbb{Z})$ is the nonabelian free group on 2 generators. Hence $SL_2(\mathbb{Z})$ is not adorable. Also by Stallings' theorem if the fundamental group of a compact 3-manifold has finitely generated nonabelian commutator subgroup which is not isomorphic to the Klein bottle group with infinite cyclic abelianization then the group is not adorable. It is known that most of these 3-manifolds support hyperbolic metric by Thurston. It is easy to show

that the pure braid group is not adorable as there is a surjection of any pure braid group of more than 2 strings onto a nonabelian free group.

From now on whenever we give examples of nonadorable groups we will mention its close relationship with nonpositively curved Riemannian manifolds. This will help us to state a general conjecture (Conjecture (0.1)).

The next result gives some important classes of examples of nonadorable groups which are generalized free products $G_1 *_H G_2$ or HNN-extensions $K*_H$. We always assume $G_1 \neq H \neq G_2$ and $K \neq H$.

Theorem (2.3). *Let G be a group.*
If $G = G_1 *_H G_2$ is a generalized free product and $G^1 \cap H = (1)$, then one of the following holds.

- G^1 *is perfect.*

- G^1 *is isomorphic to the infinite dihedral group.*

- G *is not adorable.*

If $G = K*_H = \langle K, t \ \ tHt^{-1} = \phi(H) \rangle$ is an HNN-extension and $G^1 \cap H = (1)$, then G is not adorable.
In the second case and in the last possibility of the first case for $i \geq 1$ the rank of G^i/G^{i+1} is ≥ 2.

In Corollary (2.7) we deduce a more general version of Theorem (2.3) and show that if H is n-step G-solvable (see Definition (2.6)) then in the amalgamated free product case either G is adorable of degree at most $n + 1$ or is not adorable and in the HNN-extension case it is always nonadorable.

We will give some more examples (Lemma (2.8) and Example (2.9)) of a class of nonadorable generalized free products and examples of compact Haken 3-manifolds with nonadorable fundamental groups.

At this point, recall that if M is a connected, closed oriented 3-manifold and $\pi_2(M, x) \neq 0$ then by Sphere theorem ([[9], p. 40]) there is an embedded 2-sphere in M representing a nonzero element of $\pi_2(M, x)$. Hence M can be written as a connected sum of two nonsimply connected 3-manifolds and thus $\pi_1(M, x)$ is a nontrivial free product. In addition if we assume that $\pi_1(M, x)$ is not perfect and M is not the connected sum of two projective 3-spaces then by Theorem (2.3) $\pi_1(M, x)$ is not adorable. Thus we see that most closed 3-manifolds with $\pi_2(M, x) \neq 0$ have nonadorable fundamental groups.

The next result is about groups with some geometric assumption. Recall that a torsion free Bieberbach groups is the fundamental group of a Riemannian manifold with sectional curvature equal to 0 everywhere.

Corollary (4.3). *A torsion free Bieberbach group is nonadorable unless it is solvable.*

The following Theorem is dealing with groups under some homological hypothesis. This theorem has an interesting application in knot theory and possibly in 3-manifolds in general also.

Theorem (4.4). *Let G be a group satisfying the following properties.*

- *$H_1(G, \mathbb{Z})$ has rank ≥ 3.*

- *$H_2(G^j, \mathbb{Z}) = 0$ for $j \geq 0$.*

Then G is not adorable. Moreover, G^j/G^{j+1} has rank ≥ 3 for each $j \geq 1$.

The Proposition below is a consequence of the above Theorem.

Proposition (4.7). *A knot group is adorable if and only if it has trivial Alexander polynomial.*

In fact in this case the commutator subgroup of the knot group is perfect. All other knot groups are not adorable. On the other hand any knot complement supports a complete nonpositively curved Riemannian metric ([13]).

After seeing the preprint ([17]) Tim Cochran informed me that the Proposition (4.7) was also observed by him in [[5], corollary 4.8].

Note that most of the torsion free examples of nonadorable groups we mentioned above act freely and properly discontinuously (except the braid groups case, which is still an open question) on a simply connected complete nonpositively curved Riemannian manifold. Also we recall that a solvable subgroup of the fundamental group of a nonpositively curved manifold is virtually abelian ([20]). There are generalization of these results to the case of locally $CAT(0)$ spaces ([1]). Considering these facts we pose the following conjecture.

Conjecture (0.1). *Fundamental group of generic class of complete nonpositively curved Riemannian manifolds or more generally of generic class of locally $CAT(0)$ metric spaces are not adorable.*

One can even ask the same question for hyperbolic groups.

Now we state the conjecture we referred before. Though in [17] this conjecture was stated for any finitely presented torsion free groups our primary aim was the following particular case.

Conjecture (0.2). *Let G be a finitely presented torsion free group which is isomorphic to the fundamental group of a closed aspherical 3-manifold such that G^i/G^{i+1} is a finite group for all i. Then G is adorable.*

Using Theorem (3.1) in Section 3 it is easy to show that the above conjecture is true for aspherical Seifert fibered spaces. In fact we will show that most Seifert fibered spaces have nonadorable fundamental groups.

Here note that a partial converse of the above conjecture is true for closed 3-manifold. Before we prove this claim note that the hypothesis of the conjecture implies that each G^i is finitely generated.

Lemma (0.3). *Let G be the fundamental group of a closed 3-manifold such that for some i, G^i is nontrivial, finitely generated and perfect. Then for each i, G^i/G^{i+1} is a finite group.*

Proof. Since G^i is a nontrivial perfect group, it is not a surface group. Also since G^i is finitely generated, by [[9], theorem 11.1] G^i is of finite index in G. This proves the Lemma.   □

Remark (0.4). After seeing the preprint [17] Peter A. Linnell pointed out to me that certain finite index subgroups of $SL(n, \mathbb{Z})$ for $n \geq 3$ satisfy the hypothesis of [[17], conjecture 0.2] but they are not adorable. These are some noncocompact lattices in $SL(n, \mathbb{R})$ which are residually finite p-groups and satisfy Kazhdan property T. I thank Professor Linnell for the stimulating example. We describe his example in the Appendix. Conjecture (0.2) remains open for the fundamental groups of closed aspherical 3-manifolds and for cocompact discrete subgroups of Lie groups. Considering this situation we state our main problem below.

Main Problem. *Find groups for which the Conjecture 0.2 is true.*

Note that G^i/G^{i+1} is finite for each i if and only if G/G^i is finite for each i. Thus, in other words the above conjecture says that a nonadorable aspherical 3-manifold group has an infinite solvable quotient. Compare this observation with Proposition (4.1).

Also note that by Theorem (2.3), if the group G in Conjecture (0.2) is not perfect and not isomorphic to $\mathbb{Z}_2 * \mathbb{Z}_2$ then it is irreducible. Thus we can assume that the group G in the Conjecture is irreducible. Recall that a group is *irreducible* if the group is not isomorphic to free product of two nontrivial groups.

There is another consequence of this conjecture. That is, if Conjecture (0.2) is true then the virtual Betti number conjecture will be true if a modified (half) version of it is true. We mention it below.

Modified virtual Betti number conjecture. *Let M be a closed aspherical 3-manifold such that $H_1(M, \mathbb{Z}) = 0$. Then there is a finite sheeted covering $\tilde{M}$ of M with $H_1(\tilde{M}, \mathbb{Z})$ infinite.*

It is easy to see that the Conjecture (0.2) and the Modified virtual Betti number conjecture together implies the virtual Betti number conjecture.

Virtual Betti number conjecture. *Any closed aspherical 3-manifold has a finite sheeted covering with infinite first homology group.*

The virtual Betti number conjecture was raised as a question by John Hempel in [[10], question 1.2].

1 Some elementary facts about adorable groups

Recall that a group is called *perfect* if the commutator subgroup of the group is the whole group.

Lemma (1.1). *Let $f : G \to H$ be a surjective homomorphism with G adorable. Then H is also adorable and $doa(H) \le doa(G)$.*

Example (1.2). The Artin pure braid group on more than 2 strings is not adorable, for it has a quotient a nonabelian free group. In fact the full braid group on n-strings is not adorable for $n \le 4$ and adorable of degree 1 otherwise. (see [8]).

Lemma (1.3). *The product $G \times H$ of two groups are adorable if and only if both the groups G and H are adorable. Also if $G \times H$ is adorable then $doa(G \times H) = max\{doa(G), doa(H)\}$.*

On the contrary, in the case of free product of groups, almost all the time the output is nonadorable. Hence, adorability is mainly a property for irreducible groups. We will consider the case of free product and more generally the generalized free product case in the next section.

Lemma (1.4). *Let G be an adorable group and H is a normal subgroup of G. Assume that for some i_0, G^{i_0} is simple. Then H is also adorable and $doa(H) \le doa(G)$.*

Remark (1.5). In the above lemma instead of assuming the strong hypothesis that G^{i_0} is simple we can assume only that G^{i_0} is perfect and G^{i_0}/H^{i_0+1} does not have any proper normal abelian subgroup. With this weaker hypothesis the proof follows from the fact that the kernel of the surjective homomorphism $G^{i_0}/H^{i_0+1} \to G^{i_0}/H^{i_0}$ is either trivial or $G^{i_0} = H^{i_0}$. In either case it follows that H is adorable.

Lemma (1.6). *Let H be a normal subgroup of a group G such that G^i/H^i is abelian for some i. Then G is adorable if and only if H is adorable.*

Proposition (1.7). *Let H be a normal subgroup of a group G such that G/H is solvable. Then H is adorable if and only if so is G.*

Proof. Before we start the proof, we note down some generality. Suppose G has a filtration of the form $G_n < G_{n-1} < \cdots < G_1 < G_0 = G$ where G_i is normal in G_{i-1} and G_{i-1}/G_i is abelian for each i. Since G_{i-1}/G_i is abelian for each i, we have $G'_{i-1} \subset G_i$. Replacing i by $i+1$ we get $G'_i \subset G_{i+1}$. Consequently, $G_0^i = G^i = \{G'\}^{i-1} \subset G_1{}^{i-1} \subset \{G_1{}'\}^{i-2} \subset G_2{}^{i-2} \subset \cdots \subset G'_{i-1} \subset G_i$. Thus we get $G^n \subset G_n$.

Denote G/H by F. As F is solvable we have $1 \subset F^k \subset \cdots \subset F^1 \subset F^0 = F$ where F^k is abelian. Let $\pi : G \to G/H$ be the quotient map. We have the following sequence of normal subgroups of G.

$$\cdots \subset H^n \subset H^{n-1} \cdots \subset H^1 \subset H \subset \pi^{-1}(F^k) \cdots \subset \pi^{-1}(F^0) = G.$$

Note that this sequence of normal subgroups satisfy the same properties as those of the filtration G_i of G above. Hence $G^{k+i} \subset H^{i-1}$. Now if G is adorable then for some i, G^{k+i} is perfect. We have

$$H^{k+i} \subset G^{k+i} = G^{k+k+i+2} \subset H^{k+i+1}.$$

But we already have $H^{k+i+1} \subset H^{k+i}$. That is H^{k+i} is perfect, hence H is adorable. Conversely if H is adorable then for some i, H^i is perfect. Note from the above inclusions that $H^i = G^i$ for some large i. Hence G is also adorable. $\square$

Theorem (1.8). *A group G is adorable if and only if there is a filtration $G_n < G_{n-1} < \cdots < G_1 < G_0 = G$ of G so that G_i is normal in G_{i-1}, G_{i-1}/G_i is abelian for each i and G_n is a perfect group.*

Proof. We use Proposition (1.7) and induction on n to prove the 'if' part of the Theorem. So assume that there is a filtration of G as in the hypothesis.

Then G_n is an adorable subgroup of G with solvable quotient G/G_n. Proposition (1.7) proves this implication. The 'only if' part of the Theorem follows from the definition of adorable groups. □

Corollary (1.9). *Let G be a torsion free infinite group and F be a finite quotient of G with kernel H such that H is free abelian and also central in G. Then G is adorable.*

Proof. Recall that equivalence classes of extensions of F by H are in one to one correspondence with $H^2(F, H)$ which is isomorphic to $Hom(F, (\mathbb{R}/\mathbb{Z})^n)$ where n is the rank of H ([[3], p. 95, exercise 3]). If F is perfect then $Hom(F, (\mathbb{R}/\mathbb{Z})^n) = 0$ and hence the extensions $1 \to H \to G \to F \to 1$ splits. But by hypothesis G is torsion free. Hence F is not perfect. By a similar argument it can be shown that F^i is perfect for no i unless it is the trivial group. Since F is finite this proves that F is solvable and hence G is adorable, in fact solvable. □

We sum up the above Lemmas and Propositions in the following Theorem.

Theorem (1.10). *Let H be a normal subgroup of an adorable group G. Then H is adorable if one of the following conditions is satisfied.*

- *G/H is solvable.*

- *for some i, G^i/H^i is abelian.*

- *for some i, G^i is simple.*

- *for some i, G^i is perfect and the group G^i/H^{i+1} does not have any proper abelian normal subgroup.*

Remark (1.11). It is known that any countable group is a subgroup of a countable simple group (see [[14], chapter IV, theorem 3.4]). Also we mentioned before that even finite index normal subgroup of an adorable group need not be adorable. So the above theorem is best possible in this regard.

In the next section we give some more examples of virtually adorable groups which are not adorable.

The following is an analogue of a theorem of Hirsch for poly-cyclic groups. The proofs of Lemmas (A) and (B) in the proof of the theorem are easy and we leave it to the reader.

Theorem (1.12). *The following are equivalent.*

- *G is a group which admits a filtration $G = G_0 > G_1 > \cdots > G_n$ with the property that each G_{i+1} is normal in G_i with quotient G_i/G_{i+1} cyclic and G_n is a perfect group which satisfies the maximal condition for subgroups.*

- *G is adorable and it satisfies the maximal condition for subgroups, i.e., for any sequence $H_1 < H_2 < \cdots$ of subgroups of G there is an i such that $H_i = H_{i+1} = \cdots$.*

Proof. The proof is on the same line as Hirsch's theorem. The main lemma is the following.

Lemma (A). *Let H_1 and H_2 be two subgroup of a group G and $H_1 \subset H_2$. Let H be a normal subgroup of G with the property that $H \cap H_1 = H \cap H_2$ and the subgroup generated by H and H_1 is equal to the subgroup generated by H and H_2. Then $H_1 = H_2$.*

(1) implies (2): By Theorem (1.11) it follows that (1) implies G is adorable. Now we check the maximal condition by induction on n. As G_n already satisfy maximal condition we only need to check that G_{n-1} also satisfy maximal condition which follows from the following Lemma and by noting that G_{n-1}/G_n is cyclic.

Lemma (B). *Let H be a normal subgroup of a group G such that both H and G/H satisfy the maximal condition then G also satisfies the maximal condition.*

Proof. Let $K_1 < K_2 < \cdots$ be an increasing sequence of subgroups of G. Consider the two sequences of subgroups $H \cap K_1 < H \cap K_2 < \cdots$ and $\{H, K_1\} < \{H, K_2\} < \cdots$. Here $\{A, B\}$ denotes the subgroup generated by the subgroups A and B. As H and G/H both satisfy the maximal condition there are integers k and l so that $H \cap K_k = H \cap K_{k+1} = \cdots$ and $\{H, K_l\} = \{H, K_{l+1}\} = \cdots$. Assume $k \geq l$. Then by Lemma (A) $K_k = K_{k+1} = \cdots$. $\square$

(2) implies (1): As G is adorable it has a filtration $G = G_0 > G_1 > \cdots > G_n$ with G_n perfect and each quotient abelian. Also G_n satisfies maximal condition as it is a subgroup of G and G satisfies maximal condition. Since G satisfies maximal condition each quotient G_i/G_{i+1} is finitely generated. Now a filtration as in (1) can easily be constructed.

This proves the theorem. $\square$

2   Generalized free products and adorable groups

We begin this section with the following result on free product of groups.

Recall that the infinite dihedral group D_∞ is isomorphic to $\mathbb{Z} \rtimes \mathbb{Z}_2 \simeq \mathbb{Z}_2 * \mathbb{Z}_2$.

Proposition (2.1). *The free product G of two nontrivial groups, one of which is not perfect, is either isomorphic to D_∞ or not adorable. Moreover, in the nonadorable case the rank of the abelian group G^i/G^{i+1} is greater or equal to 2 for all $i \geq 1$.*

Proof. Let G be the free product of the two nontrivial groups G_1 and G_2 and one of G_1 and G_2 is not perfect. Then, as the abelianization of $G = G_1 * G_2$ is isomorphic to $G_1/G_1^1 \oplus G_2/G_2^1$, G is also not perfect.

By Kurosh Subgroup theorem ([[14], proposition 3.6]) any subgroup of G is isomorphic to a free product $*_i A_i * F$, where F is a free group and the groups A_i are conjugates of subgroups of either G_1 or G_2. In particular the commutator subgroup G^1 is isomorphic to $*_i A_i * F$ for some A_i and F. Note that $[G_1, G_2] = \langle g_1 g_2 g_1^{-1} g_2^{-1} \mid g_i \in G_i, i = 1, 2 \rangle$ is a subgroup of G^1. Now assume that G is not D_∞. Then $[G_1, G_2]$ is a nonabelian free group and clearly $[G_1, G_2] \cap G_1 = (1) = [G_1, G_2] \cap G_2$. Also $[G_1, G_2]$ is not conjugate to any subgroup of G_1 or G_2. Hence $[G_1, G_2]$ is a subgroup of F, which shows that F is a nontrivial nonabelian free group. Hence the abelianization of G^1 is nontrivial. By a similar argument using Kurosh Subgroup theorem we conclude that no G^n is perfect. This proves the first assertion of the Proposition. The second part follows from the fact that the free group F has rank ≥ 2 and a nonabelian free group has derived series consisting of nonabelian free groups. $\square$

Remark (2.2). In Proposition (2.1) we have seen that the free product of any nontrivial group with a nonperfect group is either D_∞ or nonadorable group. The natural question arises here is what happens in the amalgamated free product case of two groups along a nontrivial group or in the case of HNN-extension? At first recall that there are examples of simple groups which are amalgamated free product of two nonabelian free groups along a (free) subgroup (see [2]). We give another example. Let $M = \mathbb{S}^3 - N(k)$ be a knot complement of a knot k in the 3-sphere. Assume that the Alexander polynomial of the knot k is nontrivial. Then by Proposition (4.7) we know that $\pi_1(M)$ is not adorable. Recall that the first homology of M is generated

by a meridian of the torus boundary of M and the longitude which is parallel to the knot in $\mathbb{S}^3$ represents the zero in $H_1(M, \mathbb{Z})$. Now glue two copies of M along the boundary which sends the above longitude of one copy to the meridian of the other and vice versa. Then the resulting manifold N has fundamental group isomorphic to the amalgamated free product $\pi_1(M) *_{\mathbb{Z} \times \mathbb{Z}} \pi_1(M)$ and an application of Mayer-Vietoris sequence for integral homology shows that N has trivial first homology. That is N has perfect fundamental group. Another example in this connection is the fundamental group of a torus knot complement in $\mathbb{S}^3$. This group is of the form $G = \mathbb{Z} *_{\mathbb{Z}} \mathbb{Z}$. If the knot is of type (p, q) then the two inclusions of $\mathbb{Z}$ in $\mathbb{Z}$ in the above amalgamated free product are defined by multiplication by p and q respectively. But G is not adorable as it has nonabelian free commutator subgroup. In the following theorem we consider a more general situation.

From now on, whenever we consider generalized free product $G = G_1 *_H G_2$ or HNN-extension $G = K*_H$, unless otherwise stated, we always assume that $G_1 \neq H \neq G_2$ and $K \neq H$.

Theorem (2.3). *Let G be a group.*
If $G = G_1 *_H G_2$ is a generalized free product and $G^1 \cap H = (1)$, then one of the following holds.

- G^1 *is perfect.*

- G^1 *is isomorphic to the infinite dihedral group D_∞.*

- G *is not adorable.*

If $G = K*_H = \langle K, t \mid tHt^{-1} = \phi(H) \rangle$ is an HNN-extension and $G^1 \cap H = (1)$, then G is not adorable.
In the second case and in the last possibility of the first case for $i \geq 1$ the rank of G^i/G^{i+1} is ≥ 2.

Note that the assumption $G^1 \cap H = (1)$ implies that H is abelian.
To prove the theorem we need to recall the bipolar structure on generalized free product and the characterization of generalized free product by the existence of a bipolar structure on the group by Stallings.

Definition (2.4). ([[14], p. 207, definition]) A *bipolar structure* on a group G is a partition of G into five disjoint subsets $H, EE, EE^*, E^*E, E^*E^*$ satisfying the following axioms. (The letters X, Y, Z will stand for the letters E or E^* with the convention that $(X^*)^* = X$, etc.)

- H is a subgroup of G.

- If $h \in H$ and $g \in XY$, then $hg \in XY$.

- If $g \in XY$, then $g^{-1} \in YX$. (*Inverse axiom*)

- If $g \in XY$ and $f \in Y^*Z$, then $gf \in XZ$. (*Product axiom*)

- If $g \in G$, there is an integer $N(g)$ such that, if there exist $g_1, \ldots, g_n \in G$ and $X_0, \ldots, X_n$ with $g_i \in X_{i-1}^* X_i$ and $g = g_1 \cdots g_n$, then $n \leq N(g)$. (*Boundedness axiom*)

- $EE^* \neq \emptyset$. (*Nontriviality axiom*)

It can be shown that every amalgamated free product or HNN-extension has a bipolar structure [[14], p. 207-208]. The following theorem of Stallings shows that the converse is also true.

Theorem (2.5). ([[14], theorem 6.5]) *A group G has a bipolar structure if and only if G is either a nontrivial free product with amalgamation (possibly an ordinary free product) or an HNN-extension.*

Proof of Theorem (2.3). At first note that the first 5 properties in the above definition are hereditary, that is any subgroup F of G has a partition by subsets satisfying these properties. The induced partition of F is obtained by taking intersection of $H, EE, \ldots$ with F. But $EE^* \cap F$ could be empty. We replace F by the commutator subgroup G^1 of G. We would like to check the sixth property (that is, the nontriviality axiom) for this induced partition on G^1.

We consider the amalgamated free product case first. Recall that if we write $g \in G - H$ in the form $g = c_1 \cdots c_n$ where no $c_i \in H$ and each c_i is in one of the factors G_1 or G_2 and successive c_i, c_{i+1} come from different factors, then $g \in EE^*$ if and only if $c_1 \in G_1$ and $c_n \in G_2$. Such a word is called *cyclically reduced*. Thus EE^* consists of all cyclically reduced words. Let $g_1 \in G_1 - H$ and $g_2 \in G_2 - H$, then $g_1 g_2 g_1^{-1} g_2^{-1}$ is a cyclically reduced word and is contained in $EE^* \cap G^1$. Hence the induced partition on G^1 defines a bipolar structure on G^1 with amalgamating subgroup $G^1 \cap H = (1)$. Hence G^1 is a free product of two nontrivial groups. Using Proposition (2.1) we complete the proof in this case.

When G is an HNN-extension we have a similar situation. We have to check that $EE^* \cap G^1 \neq \emptyset$. Recall from [[14], p. 208] that if we write

$g \in G - H$ in the reduced form $g = h_0 t^{\epsilon_1} h_1 \cdots t^{\epsilon_n} h_n$ (where $\epsilon_i = \pm 1$ and $h_i \in K$ for each i) then $g \in EE^*$ if and only if $h_0 \in K - H$, or $h_o \in H$ and $\epsilon = +1$, and $h_n \in H$ and $\epsilon_n = +1$. Now let $h_0 \in K - H$ and $h_1 \in H$, then $h_0(h_1 t^{-1}) h_0^{-1} (h_1 t^{-1})^{-1} = (h_0 h_1) t^{-1} h_0^{-1} t h_1^{-1} \in EE^* \cap G^1$. Hence the induced partition on G^1 gives a bipolar structure on G^1. Since $G^1 \cap H = (1)$ we get that G^1 is a free product of a nontrivial group with the infinite cyclic group. Hence Proposition (2.1) applies again.   □

We introduce below a stronger version of the notion of solvability which depends both on the group and the group where it is embedded.

Definition (2.6). A subgroup H of a group G is called *G-solvable* (or *subgroup solvable*) if $G^n \cap H = (1)$ for some n. If in addition $G^{n-1} \cap H \neq (1)$ then H is called *n-step G-solvable* (or *n-step subgroup solvable*).

Note that if H is G-solvable then H is solvable. Also if G is solvable then any subgroup of G is G-solvable.

Now we can state a Corollary of Theorem (2.3). The proof is easily deduced from the proof of Theorem (2.3) and is left to the reader.

Corollary (2.7). *Let G be a group.*
If $G = G_1 *_H G_2$ is a generalized free product and H is n-step G-solvable, then one of the following holds.

- *G is adorable of degree n and not solvable.*

- *$G^n \simeq D_\infty$.*

- *G is not adorable.*

If $G = K*_H = \langle K, t \quad tHt^{-1} = \phi(H) \rangle$ is an HNN-extension and H is G-solvable, then G is not adorable.

In the second case and in the last possibility of the first case for $i \geq 1$ the rank of G^i/G^{i+1} is ≥ 2.

The following Lemma consider some more generalized free product cases.

Lemma (2.8). *Let $G_1 *_H G_2$ be a generalized free product with H abelian and is contained in the center of both G_1 and G_2. Also assume that one of G_1/H or G_2/H is not perfect. Then $G_1 *_H G_2$ is either solvable or not adorable.*

Proof. Using normal form of elements of $G_1 *_H G_2$ it is easy to show that the center of $G_1 *_H G_2$ is H. This implies that we have a surjective homomorphism $G_1 *_H G_2 \to (G_1 *_H G_2)/H = G_1/H * G_2/H$. By Proposition (2.1) $G_1/H *$

G_2/H is either the infinite dihedral group or not adorable and hence $G_1 *_H G_2$ is either solvable or not adorable by Lemma (1.1). □

Example (2.9). Using Lemma (2.8) we now give a large class of examples of compact Haken 3-manifolds with nonadorable fundamental groups. Let M and N be two compact orientable Seifert fibered 3-manifolds with nonempty boundary and orientable base orbifold. Such examples of M and N are torus knot complements in $\mathbb{S}^3$. Let ∂M and ∂N be the boundary components of M and N respectively. Note that both ∂M and ∂N are tori. Let $\gamma_1 \subset \partial M$ and $\gamma_2 \subset \partial N$ be simple closed curves which are parallel to some regular fiber of M and N respectively. Recall that both γ_1 and γ_2 represent central elements of $\pi_1(M)$ and $\pi_1(N)$ respectively. Now choose an annulus neighborhood A_1 of γ_1 in ∂M and A_2 of γ_2 in ∂N and glue M and N identifying A_1 with A_2 by a diffeomorphism which sends γ_1 to γ_2. Let P be the resulting manifold. Then P is a compact Haken 3-manifold with tori boundary and by Seifert-van Kampen theorem $\pi_1(P)$ satisfies the hypothesis of Lemma (2.8) and hence either solvable or not adorable. Here note that the manifold P itself is Seifert fibered. In the next section we will show that in fact an infinite group which is the fundamental group of a compact Seifert fibered 3-manifold is nonadorable except for some few cases.

3   Adorability and 3-manifolds

Seifert fibered spaces are a fundamental and very important class of 3-manifolds. Conjecturally (due to Thurston) any 3-manifold is build from Seifert fibered spaces and hyperbolic 3-manifolds. Results of Jaco-Shalen, Johannson and Thurston say that this is in fact true for any Haken 3-manifold.

Theorem (3.1). *Let M^3 be a compact Seifert fibered 3-manifold. Then one of the following four cases occur.*

- *$(\pi_1(M))^i$ is finite for some $i \leq 2$.*

- *$\pi_1(M)$ is solvable.*

- *$\pi_1(M)$ is not adorable and $(\pi_1(M))^i/(\pi_1(M))^{i+1}$ has rank greater than 1 for all i greater than some i_0.*

- *$\pi_1(M)$ is perfect.*

Proof. At first we recall some well known group theoretic informations about the fundamental group of Seifert fibered spaces. If B is the base orbifold of M then there is a surjective homomorphism $\pi_1(M) \to \pi_1^{orb}(B)$, where $\pi_1^{orb}(B)$ is the orbifold fundamental group of B. Recall that $\pi_1^{orb}(B)$ is a Fuchsian group. Also recall that the above surjective homomorphism is part of the following exact sequence.

$$1 \to \langle t \rangle \to \pi_1(M) \to \pi_1^{orb}(B) \to 1.$$

Here $\langle t \rangle$ is the cyclic normal subgroup of $\pi_1(M)$ generated by a regular fiber of the Seifert fibration of M. Also if $\pi_1(M)$ is infinite then $\langle t \rangle$ is an infinite cyclic subgroup of $\pi_1(M)$.

Some examples of Seifert fibered 3-manifolds with finite fundamental group are lens spaces and the Poincare sphere. So, from now on we assume $\pi_1(M)$ is infinite. Then there is an exact sequence.

$$1 \to \mathbb{Z} \to \pi_1(M) \to \pi_1^{orb}(B) \to 1.$$

There are now two cases to consider.

Case 1. $\pi_1^{orb}(B)$ is finite. By [[7], lemma 2.5] $\pi_1(M)$ has a finite normal subgroup G with quotient isomorphic either to $\mathbb{Z}$ or to D_∞. Since D_∞ is solvable $(\pi_1(M))^i$ is finite for some $i \leq 2$.

Case 2. $\pi_1^{orb}(B)$ is infinite and not a perfect group. Then by [[18], theorem 1.5] there is a torsion free normal subgroup H of $\pi_1^{orb}(B)$ so that $\pi_1^{orb}(B)/H$ is a finite solvable group. Hence by Proposition (1.7) $\pi_1^{orb}(B)$ is adorable if and only if so is H. Since H is of finite index in $\pi_1^{orb}(B)$ by a result of Hoare, Karrass and Solitar [[14], chapter III, proposition 7.4] H is again a Fuchsian group. But a torsion free Fuchsian group is the fundamental group of a compact surface (evident from the presentation of such groups). Hence H is either $\mathbb{Z}$ or $\mathbb{Z} \times \mathbb{Z}$ or $\mathbb{Z} \rtimes \mathbb{Z}$ or nonadorable. Thus by Proposition (1.7) $\pi_1^{orb}(B)$ is either solvable or nonadorable. If $\pi_1^{orb}(B)$ is solvable then from the above exact sequence it follows that $\pi_1(M)$ is also solvable. On the other hand Lemma (1.1) shows $\pi_1(M)$ is nonadorable whenever $\pi_1^{orb}(B)$ is.

Next, consider the case when $\pi_1^{orb}(B)$ is a perfect group. Let $x_1, x_2, \ldots, x_n$ be the cone points on B with indices $p_1, p_2, \ldots, p_n$ greater than or equal to 2. By [[18], theorem 1.5] $\pi_1^{orb}(B)$ is perfect if and only if $B = \mathbb{S}^2$ and the indices $p_1, p_2, \ldots, p_n$ are pairwise coprime. It is well known that in this situation M is an integral homology 3-sphere and hence $\pi_1(M)$ is also perfect. This proves the theorem. $\square$

Notice that the proof of the above theorem is not very illuminating in the sense that it does not show the cases when the groups are nonadorable or

solvable. Below we show that in fact in most cases the fundamental group of a compact Seifert fibered space is nonadorable. For simplicity of presentation we consider Seifert fibered spaces whose base orbifold B is orientable and has only cone singularities. Note that the proof of the Theorem deals with both orientable and nonorientable B and for any kind of singularities. At first let us consider the case when M has nonempty boundary. Since B also has nonempty boundary, $\pi_1^{orb}(B)$ is a free product of cyclic groups ([9]) and hence by Proposition (2.1) $\pi_1^{orb}(B)$ is either the infinite dihedral group or is nonadorable if it is a nontrivial free product. Hence either $\pi_1(M)$ is solvable (when $\pi_1^{orb}(B)$ is dihedral or cyclic) or (by Lemma (1.1)) $\pi_1(M)$ is not adorable.

If M is closed then we have the same situation as above except that $\pi_1^{orb}(B)$ has the following form.

$$\pi_1^{orb}(B) = \langle a_1, \ldots, a_g, b_1, \ldots, b_g, x_1, \ldots, x_n \mid x_1^{j_1} = \cdots = x_n^{j_n} = 1;$$

$$\Pi_{j=1}^g [a_j, b_j] x_1 \cdots x_n = 1 \rangle$$

where $x_1, \ldots x_n$ represents loops around cone points of B. We will consider the case $g = 0$ at the end of the proof. If $g \geq 1$ then adding the extra relations $a_1 = 1$ we get that $\pi_1^{orb}(B)$ has the following homomorphic image

$$\langle a_2, \ldots, a_g, b_1, \ldots, b_g, x_1, \ldots, x_n \mid x_1^{j_1} = \cdots = x_n^{j_n} = 1;$$

$$\Pi_{j=2}^g [a_j, b_j] x_1 \cdots x_n = 1 \rangle.$$

If there is no cone point on B and $g = 1$ then M is an $\mathbb{S}^1$-bundle over the torus and hence has solvable fundamental group. Otherwise the last group is a free product of the infinite cyclic group (generated by b_1) and another group and hence not adorable by Proposition (2.1). Thus $\pi_1^{orb}(B)$ is also not adorable by Lemma (1.1). Consequently so is $\pi_1(M)$.

Now we consider the case when $g = 0$. There are further two cases to consider.

Case A. $\pi_1^{orb}(B)$ is finite. This case occurs when B has at most 3 cone points and if exactly 3 cone points with indices n_1, n_2, n_3 then $\frac{1}{n_1} + \frac{1}{n_2} + \frac{1}{n_3} > 1$ (see [[9], theorem 12.2]). We have already discussed this case in **Case 1** in the proof of the theorem.

Case B. $\pi_1^{orb}(B)$ is infinite. In this case there are the following two possibilities (see [[9], theorem 12.2]). (a) B has more than 3 cone points. (b) B has 3 cone points with indices j_1, j_2, j_3 so that $\frac{1}{j_1} + \frac{1}{j_2} + \frac{1}{j_3} \leq 1$.

For (a) we need the following easily verified remark.

Remark (3.2). If B is a sphere with 3 cone points then $|\pi_1^{orb}(B)| \geq 3$.

Now recall that in (a) $\pi_1^{orb}(B)$ has the following presentation.

$$\langle x_1, \ldots, x_n \mid x_1^{j_1} = \cdots = x_n^{j_n} = 1; x_1 \cdots x_n = 1 \rangle$$

where $n \geq 4$. Now assume $n \geq 6$ and add the relation $x_1 x_2 x_3 = 1$ in the above presentation. Then $\pi_1^{orb}(B)$ surjects onto the free product of

$$\langle x_1, x_2, x_3 \mid x_1^{j_1} = x_2^{j_2} = x_3^{j_3} = 1; x_1 x_2 x_3 = 1 \rangle$$

and

$$\langle x_4, \ldots, x_n \mid x_4^{j_4} = \cdots = x_n^{j_n} = 1; x_4 \cdots x_n = 1 \rangle.$$

By Proposition (2.1) and Remark (3.2) it follows that $\pi_1^{orb}(B)$ is either perfect or not adorable and hence so is $\pi_1(M)$. In the case $n = 5$ if there is a pair of indices j_k and j_l so that $(j_k, j_l) \geq 3$ then it is easy to show that $\pi_1(M)$ is nonadorable. We leave the remaining cases to the reader.

In (b) when $\frac{1}{j_1} + \frac{1}{j_2} + \frac{1}{j_3} = 1$ then $\pi_1^{orb}(B)$ is a discrete group of isometries of the Euclidean plane. Recall that a torsion free discrete group of isometries of the Euclidean plane is isomorphic to $\mathbb{Z}$ or $\mathbb{Z} \times \mathbb{Z}$ or $\mathbb{Z} \rtimes \mathbb{Z}$ and hence by the result of Sah we mentioned above $\pi_1^{orb}(B)$ is either perfect or solvable. On the other hand if $\frac{1}{j_1} + \frac{1}{j_2} + \frac{1}{j_3} < 1$ then $\pi_1^{orb}(B)$ is a discrete groups of isometries of the hyperbolic plane. Since a group of isometries of the hyperbolic plane does not contain a free abelian group on more than one generator, it follows by the result of Sah that in this case $\pi_1^{orb}(B)$ is either perfect or a finite solvable extension of $\mathbb{Z}$ or nonadorable. Hence $\pi_1(M)$ is either solvable or perfect or nonadorable.

Remark (3.3). Recall that a Fuchsian group is a discrete subgroup of $PSL(2, \mathbb{R})$ and it is either a free product of cyclic groups or is isomorphic to a group of the form $\pi_1^{orb}(B)$. In the free product case except for the infinite dihedral group all other free products are nonadorable. In the remaining cases we have already seen in the proof of Theorem (3.1) that a Fuchsian group is either finite or perfect or solvable or nonadorable and in most cases it is nonadorable. It is not known to me if a similar situation occur for discrete subgroups of $PSL(2, \mathbb{C})$. Such informations will be very useful to get some hold on the virtual Betti number conjecture for hyperbolic 3-manifolds.

4 (Non)adorability under homological or geometric hypothesis

In Section 2 under some group theoretic hypothesis we showed when a generalized free product or an HNN-extension produces a nonadorable group.

This section deals with some homological or geometric (or topological) hypothesis on a group which ensures that the group is nonadorable.

Proposition (4.1). *Let M^3 be a compact 3-manifold with the property that there is an exact sequence of groups $1 \to H \to \pi_1(M) \to F \to 1$ such that H is finitely generated nonabelian but not the fundamental group of the Klein bottle and F is an infinite solvable group. Then $\pi_1(M)$ is not adorable.*

Proof. By theorem 11.1 in [9] it follows that H is the fundamental group of a compact surface. Also as H is nonabelian and not the Klein bottle group, it is not adorable. The Proposition now follows from Proposition (1.7). □

Proposition (4.2). *Let G be a torsion free group and H a free nonabelian (or abelian) normal subgroup of G with quotient F a nontrivial finite (or finite perfect) group. Then G is not adorable.*

Proof. If H is nonabelian then by Stallings' theorem G itself is free and hence not adorable. So assume H is free abelian. Since in this case F is a perfect group, the restriction of the quotient map $G \to F$ to G^i is again surjective for each i with $H \cap G^i$ as kernel. And since G is infinite and torsion free, $H \cap G^i$ is nontrivial free abelian for all i. This shows that each G^i is again a Bieberbach group. Note that if $H^1(G^i, \mathbb{Z}) = 0$ then G^i is centerless and it is known that centerless Bieberbach groups are meta-abelian with nontrivial abelian holonomy group and hence solvable ([11]). But since each G^i surjects onto a nontrivial perfect group it cannot be solvable. Hence $H^1(G^i, \mathbb{Z}) \neq 0$ for each i. This proves the Proposition. □

The conclusion of the above Proposition remains valid if we assume that F is nonsolvable adorable.

By Bieberbach theorem ([4]) we have the following Corollary.

Corollary (4.3). *The fundamental group of a closed flat Riemannian manifold is nonadorable unless it is solvable.*

So far we have given examples of nonadorable groups which are fundamental groups of known class of manifolds or of manifolds with some strong Riemannian structure. The following Theorem gives a general class of examples of nonadorable groups under some homological conditions.

Theorem (4.4). *Let G be a group satisfying the following properties.*

- $H_1(G, \mathbb{Z})$ *has rank* ≥ 3.

- $H_2(G^j, \mathbb{Z}) = 0$ *for* $j \geq 0$.

Then G is not adorable. Moreover, G^j/G^{j+1} has rank ≥ 3 for each $j \geq 1$.

Proof. Consider the short exact sequence.

$$1 \to G^1 \to G \to G/G^1 \to 1.$$

We use the Hochschild-Serre spectral sequence ([[3], p. 171]) of the above exact sequence. The E^2-term of the spectral sequence is $E^2_{pq} = H_p(G/G^1, H_q(G^1, \mathbb{Z}))$. Here $\mathbb{Z}$ is considered as a trivial G-module. This spectral sequence gives rise to the following five term exact sequence.

$$H_2(G, \mathbb{Z}) \to E^2_{20} \to E^2_{01} \to H_1(G, \mathbb{Z}) \to E^2_{10} \to 0.$$

Using (2) we get

$$0 \to H_2(G/G^1, H_0(G^1, \mathbb{Z})) \to H_0(G/G^1, H_1(G^1, \mathbb{Z})) \to H_1(G, \mathbb{Z})$$

$$\to H_1(G/G^1, H_0(G^1, \mathbb{Z})) \to 0.$$

As $\mathbb{Z}$ is a trivial G-module we get

$$0 \to H_2(G/G^1, \mathbb{Z}) \to H_0(G/G^1, H_1(G^1, \mathbb{Z})) \to H_1(G, \mathbb{Z}) \to$$

$$H_1(G/G^1, \mathbb{Z}) \to 0.$$

Note that the homomorphism between the last two nonzero terms in the above exact sequence is an isomorphism. Also the second nonzero term from left is isomorphic to the co-invariant $H_1(G^1, \mathbb{Z})_{G/G^1}$ and hence we have the following

$$H_2(G/G^1, \mathbb{Z}) \simeq H_1(G^1, \mathbb{Z})_{G/G^1}.$$

Since G/G^1 has rank ≥ 3 we get that $H_2(G/G^1, \mathbb{Z})$ has rank greater or equal to 3. This follows from the following lemma.

Lemma (4.5). *Let A be an abelian group. Then the rank of $H_2(A, \mathbb{Z})$ is $rkA(rkA - 1)/2$ if rkA is finite and infinite otherwise.*

Proof. If A is finitely generated then from the formula $H_2(A, \mathbb{Z}) \simeq \bigwedge^2 A$ it follows that rank of $H_2(A, \mathbb{Z})$ is $rkA(rkA - 1)/2$. In the case A is countable

and infinitely generated then there are finitely generated subgroups A_n of A such that A is the direct limit of A_n. Now as homology of group commutes with direct limit the proof follows using the previous case. Similar argument applies when A is uncountable. □

To complete the proof of the theorem note that there is a surjective homomorphism $H_1(G^1, \mathbb{Z}) \to H_1(G^1, \mathbb{Z})_{G/G^1}$. Thus we have proved that $H_1(G^1, \mathbb{Z})$ also has rank ≥ 3. Finally replacing G by G^n and G^1 by G^{n+1} and using induction on n the proof is completed. □

There are two important consequences of Theorem (4.4). At first we recall some definition from [19].

Let R be a nontrivial commutative ring with unity. The class $E(R)$ consists of groups G for which the trivial G-module R has a RG-projective resolution

$$\cdots \to P_2 \to P_1 \to P_0 \to R \to 0$$

such that the map $1_R \otimes \partial_2 : R \otimes_{RG} P_2 \to R \otimes_{RG} P_1$ is injective. Note that if a group belongs to $E(R)$ then $H_2(G, R) = 0$. Also this condition is sufficient to belong to $E(R)$ for groups of cohomological dimension less or equal to 2. By definition G lies in E if it belongs to $E(R)$ for all R. A characterization of E-groups is that a group G is an E-group if and only if G belongs to $E(\mathbb{Z})$ and G/G^1 is torsion free ([[19], lemma 2.3]).

Corollary (4.6). *Let G be an E-group and rank of $H_1(G, \mathbb{Z})$ is ≥ 2. Then G is not adorable.*

Proof. By [[19], theorem A] it follows that G satisfies the second condition of Theorem (4.4). Hence we get that $H_1(G^2, \mathbb{Z})$ has rank ≥ 1 and hence in particular G^2 is not perfect. On the other hand an E-groups has derived length $0, 1, 2$ or infinity (remark after [[19], theorem A]). Thus G is not adorable. □

In the following Proposition we give an application of Theorem (4.4) for knot groups.

Proposition (4.7). *Let $H = \pi_1(\mathbb{S}^3 - k)$, where k is a nontrivial knot in the 3-sphere with nontrivial Alexander polynomial. Then H is not adorable. Moreover if rank of H^1/H^2 is greater or equal to 3 then the same is true for H^j/H^{j+1} for all $j \geq 2$.*

In fact a stronger version of the Proposition follows, namely by [19] the successive quotients of the derived series of G are torsion free. Thus we get that the successive quotients of the derived series are nontrivial and torsion free.

Proof of Proposition (4.7). At first recall that the second condition of Theorem (4.4) follows from [[19], theorem A]. On the other hand the commutator subgroup of a knot group is perfect if and only if the knot has trivial Alexander polynomial. So assume that H^1 is not perfect. If H^1 is finitely generated then in fact it is nonabelian free and hence H is not adorable. If rank of H^1/H^2 is ≥ 3 then the proof follows from the above Theorem. So assume that rank of H^1/H^2 is ≤ 2.

Recall that the rank of the abelian group H^1/H^2 is equal to the degree of the Alexander polynomial of the knot (see [[6], theorem 1.1]). Thus if rank of H^1/H^2 is 1 then the Alexander polynomial has degree 1 which is impossible as the Alexander polynomial of a knot always has even degree. Next if rank of H^1/H^2 is 2 then H is not adorable by Corollary (2.6) and noting that knot groups are E-groups. $\square$

Definition (4.8). A Lie group is called *adorable* if it is adorable as an abstract group.

Theorem (4.9). *Every connected (real or complex) Lie group is adorable.*

Proof. Let G be a Lie group and consider its derived series.

$$\cdots \subset G^n \subset G^{n-1} \cdots \subset G^1 \subset G^0 = G.$$

Note that each G^i is a normal subgroup of G. Define $G_i = \overline{G^i}$. Then we have a sequence of normal subgroups

$$\cdots \subset G_n \subset G_{n-1} \cdots \subset G_1 \subset G_0 = G$$

so that G_i is a closed Lie subgroup of G and G_i/G_{i+1} is abelian for each i. Suppose for some i, $\dim G_i = 0$, i.e., G_i is a closed discrete normal subgroup of G. We claim G_i is abelian. For, fix $g_i \in G_i$ and consider the continuous map $G \to G_i$ given by $g \mapsto gg_ig^{-1}$. As G is connected and G_i is discrete image of this map is the singleton $\{g_i\}$. That is g_i commutes with all $g \in G$ and hence G_i is abelian.

As $G^i \subset G_i$, G^i is also abelian. Thus G is solvable and hence adorable.

Next assume no G_i is discrete. Then as G is finite dimensional and G_i's are Lie subgroups of G there is an i_0 so that $G_j = G_{j+1}$ for all $j \geq i_0$ and $\dim G_{i_0} \geq 1$. We need the following Lemma to complete the proof of the Theorem.

Lemma (4.10). *Let G be a (real or complex) Lie group such that $\overline{G^1} = G$. Then $G^2 = G^1$, that is G^1 is a perfect group.*

Proof. The proof of the lemma follows from [[12], theorem XII.3.1 and theorem XVI.2.1]. □

We have $G^{i_0} \subset G_{i_0}$ and hence

$$G_{i_0} = G_{i_0+1} = \overline{G^{i_0+1}} \subset \overline{G^1_{i_0}} \subset \overline{G_{i_0}} = G_{i_0}.$$

This implies $\overline{G^1_{i_0}} = G_{i_0}$. Now from the above Lemma we get G_{i_0} is adorable. Thus G_{i_0} is a normal adorable subgroup of G_{i_0-1} with quotient G_{i_0-1}/G_{i_0} abelian and hence by Proposition (1.7) G_{i_0-1} is also adorable. By induction it follows that G is adorable. □

5 Appendix

In this section we describe the counter example given by Peter A. Linnell to [[17], conjecture 0.2].

Example (5.1). (P.A. Linnell) Let $n \geq 3$ and p be an odd prime. Let K be the kernel of the homomorphism $SL(n, \mathbb{Z}) \to SL(n, \mathbb{Z}/p\mathbb{Z})$ which is induced by the homomorphism $\mathbb{Z} \to \mathbb{Z}/p\mathbb{Z}$. When $p = 2$, let K be the kernel of $SL(n, \mathbb{Z}) \to SL(n, \mathbb{Z}/4\mathbb{Z})$. Now we have the following three facts about K.

- K is a residually finite p-group. Hence we get that K^{i+1} is a proper subgroup of K^i for each i.

- K satisfies Kazhdan property T. Hence K^i/K^{i+1} is a finite group for each i.

- K is finitely presented and torsion free.

Thus K is not adorable. But by the second and the third fact above, K satisfies the hypothesis of [[17], conjecture (0.2)].

A notable fact is that K is a noncocompact discrete subgroup of $SL(n, \mathbb{R})$. It will be very interesting to prove Conjecture (0.2) for cocompact discrete subgroup of Lie groups.

6 Problems

In this section we state some problems for a further study on adorable groups. We also give the motivations behind each problem and mention known results related to the problem.

Problem (6.1). *Study the Main Problem for some particular class of groups, for example for cocompact discrete subgroups of Lie groups or for groups which are fundamental groups of closed nonpositively curved Riemannian manifolds.*

Problem (6.1) is related to the particular case of the virtual Betti number conjecture for hyperbolic 3-manifolds. We have already seen that a discrete subgroup of $PSL(2, \mathbb{R})$ is either finite or solvable or perfect or nonadorable. In fact it is possible to describe when each of these possibilities occur. A similar result about discrete subgroup of $PSL(2, \mathbb{C})$ will be very important. A more precise problem is the following.

Problem (6.2). *Given a positive integer n does there exist a discrete (torsion free) subgroup of $PSL(2, \mathbb{C})$ which is adorable of degree n?*

Problem (6.3). *Find all 3-manifolds with adorable fundamental group.*

Some examples of such 3-manifolds are integral homology 3-spheres and knot complement of knots with trivial Alexander polynomial. In Theorem (3.1) we have seen that most Seifert fibered spaces have nonadorable fundamental group and also we have shown when the fundamental group is adorable.

Problem (6.4). *Prove that most groups are not adorable.*

A possible approach to study Problem (6.4) is by the same method which was used to show that most groups are hyperbolic.

A small and first step towards Conjecture (0.2) is the following.

Problem (6.5). *Show that Conjecture (0.2) is true for the fundamental groups of compact Haken 3-manifolds.*

We have already mentioned that it is true for Seifert fibered spaces. Note that if the fundamental group of a compact Haken 3-manifold satisfies the hypothesis of Conjecture (0.2) then the manifold has to be closed.

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